**Linear and Vector Algebra**

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**1. Vector Algebra**

**1.1 Definition & Notation**

* **Vector**: An element of Rn\mathbb{R}^n (or any field extension) often denoted v=(v1,v2,...,vn)\mathbf{v} = (v\_1, v\_2, ..., v\_n).
* **Skeptical question**: Why restrict to real fields? Many properties break over finite fields or non‑commutative rings—be mindful of the ambient structure.

**1.2 Basic Operations**

1. **Addition**: u+v=(u1+v1,...,un+vn)\mathbf{u} + \mathbf{v} = (u\_1 + v\_1, ..., u\_n + v\_n).
2. **Scalar multiplication**: cv=(cv1,...,cvn)c\mathbf{v} = (cv\_1, ..., cv\_n).

**Pitfall**: You cannot "divide" vectors. If someone casually writes v/u\mathbf{v}/\mathbf{u}, they’re sloppy; always reduce to scalar multiplication or solve systems.

**1.3 Dot (Inner) Product**

* **Definition**: u⋅v=∑i=1nuivi\mathbf{u} \cdot \mathbf{v} = \sum\_{i=1}^n u\_i v\_i.
* **Geometric**: u⋅v=∥u∥∥v∥cos⁡θ\mathbf{u} \cdot \mathbf{v} = \|u\|\|v\|\cos\theta.

**Strong opinion**: Many textbooks overemphasize Euclidean intuition. In high‑dimensional data, "orthogonality" loses meaning — distances concentrate.

**1.4 Cross Product & Determinants**

* **Cross product** (only in R3\mathbb{R}^3): u×v=(u2v3−u3v2,u3v1−u1v3,u1v2−u2v1)\mathbf{u} \times \mathbf{v} = (u\_2v\_3 - u\_3v\_2, u\_3v\_1 - u\_1v\_3, u\_1v\_2 - u\_2v\_1).
* **Properties**: It’s anti‑commutative (u×v=− v×uu \times v = -\,v \times u). Output is perpendicular to both inputs.

**Caveat**: The cross product isn’t generalizable to n≠3n\ne3. Stop pretending it’s a universal tool; use wedge products in higher dims.

**1.5 Common Pitfalls & Strong Opinions**

* **"Normalized vectors always nicer"**: Normalization can amplify noise in data—use with caution.
* **Beware metric assumptions**: Dot‑product–based methods assume Euclidean geometry; in some applications (e.g., text embeddings), other metrics (cosine, Manhattan) may fare better.

**2. Linear Algebra**

**2.1 Vector Spaces & Subspaces**

* **Vector space** VV over field FF: closed under addition & scalar multiplication, with 8 axioms (associativity, commutativity, identity, inverses, distributivity, etc.).
* **Subspace** W⊆VW\subseteq V: nonempty, closed under the same operations.

**Skeptical lens**: Treat axioms not as rote rules but as the minimal requirements for linear reasoning—missing any, and your "space" might be a chaotic mess.

**2.2 Linear Combinations, Span, and Basis**

* **Linear combination**: a1v1+...+akvka\_1\mathbf{v}\_1 + ... + a\_k\mathbf{v}\_k.
* **Span**: All linear combinations of a set; span(S)\text{span}(S) is the smallest subspace containing SS.
* **Basis**: A linearly independent set that spans VV.

**Opinionated note**: Many instructors gush over "the" basis, but bases lack uniqueness. What truly matters is dimension, not your chosen coordinate axes.

**2.3 Linear Independence & Dimension**

* **Linear independence**: No nontrivial combination yields zero.
* **Dimension**: Number of vectors in any basis of VV.

**Pitfall**: Don’t conflate "independent rows" with "independent columns"—for rectangular matrices, row‐space and column‐space differ.

**2.4 Linear Transformations & Matrices**

* **Linear map** T:V→WT: V \to W: T(u+v)=T(u)+T(v)T(u+v)=T(u)+T(v), T(cu)=cT(u)T(cu)=cT(u).
* **Matrix representation** depends entirely on basis choices.
* **Matrix multiplication**: ABAB means apply BB then AA. Order matters.

**Let’s be frank**: Matrices are bookkeeping devices, not mystical objects. Don’t chase matrix aesthetics over understanding the map’s action.

**2.5 Eigenvalues & Eigenvectors**

* **Definition**: T(v)=λvT(v)=\lambda v or Av=λvAv=\lambda v.
* **Characteristic polynomial**: det⁡(A−λI)=0\det(A - \lambda I)=0.

**Caveat**: Real matrices often aren’t diagonalizable. Don’t assume every system decouples neatly—consider Jordan forms or numeric approximations.

**2.6 Rank–Nullity & Applications**

* **Theorem**: dim⁡(ker⁡T)+dim⁡(im T)=dim⁡V\dim(\ker T) + \dim(\mathrm{im}\,T) = \dim V.

**Applications**:

* **Solving Ax=bAx=b**: Consistency ⇔ b∈im(A)b\in\mathrm{im}(A).
* **Least squares**: x=arg⁡min⁡∥Ax−b∥2x=\arg\min\|Ax-b\|^2 solves ATAx=ATbA^TAx=A^Tb.
* **PCA**: Data covariance matrix eigen‐analysis. Beware centering/scaling.

**2.7 Common Pitfalls & Strong Opinions**

* **Determinant worship**: Determinants provide volume scaling, but computing them in high dims is a waste—prefer LU or SVD decompositions.
* **Blindly trusting numeric routines**: Floating‐point eigenvalues can be garbage if the matrix is ill‐conditioned; always check condition number